Crossover from inverse power law to stretched exponential for critical branched chain processes with memory

1993 J. Phys. A: Math. Gen. 26 4183

(http://iopscience.iop.org/0305-4470/26/17/026)

J. Phys. A: Math. Gen. 26 (1993) 4183–4196. Printed in the UK

Crossover from inverse power law to stretched exponential for critical branched chain processes with memory

Marcel Ovidiu Vlad†

Institut für Festkörperforschung, Forschungszentrum Jülich W-5170 Jülich, Federal Republic of Germany

Received 29 September 1992, in final form 29 March 1993

Abstract. A new type of branched chain processes is introduced, based on the assumption that at the critical state, memory effects are present and thus the probability λ_q that at the qth generation a new particle is created depends on the generation index q. This dependence may be described in terms of a kind of hierarchical clustering. The main results are the following: (a) for the supracritical regime the probability $\xi(N)$ that the offspring of an initial particle is N has an inverse power law behaviour $\xi(N)\sim N^{-(1+\mathscr{H})}\Xi(\ln N)$ as $N\to\infty$ where $\mathscr{H}>0$ is a fractal exponent related to the mean number $\bar{\nu}$ of particles produced per generation and $\Xi(\ln N)$ is a periodic function of $\ln N$; (b) for the subcritical regime $\xi(N)$ has an exponential tail $\xi(N)\sim\exp(-N\ln(1/\bar{\nu}))$; (c) for the critical regime the asymptotic behaviour is described by a stretched exponential $\xi(N)\sim$ constant $N^{-1}\exp(-KN^{1/(2-H)})$ where $1>H\geqslant 0$ is another fractal exponent describing the memory effects.

1. Introduction

Memory effects have been extensively investigated in physical and mathematical literature. Various examples from polymer physics, normal and exotic diffusion, dynamics of growth processes, diffusion limited aggregation, kinetic critical phenomena, etc., have been analysed (Bouchaud and Georges 1990, Freed 1987, Haus and Kehr 1978, 1979, Kutner 1985, Iosifescu and Grigorescu 1989, Peliti and Pietronero 1987, Pietronero and Sibesma 1987, Shlesinger and Klafter 1989). Our knowledge of memory effects comes mainly from numerical simulations, however, several analytically tractable models have also been investigated.

Within this paper we aim at giving a new theoretical description of the memory effect for critical branched chain processes. A branched chain process is a simple dynamical phenomenon which is analytically tractable. Although the modelling of a chain process is relatively simple, as far as we know no attempts to analyse the memory effects have been made.

The starting point of our approach is a renormalized theory of supracritical branched chain processes suggested by Vlad (1991). We shall try to incorporate the memory effects into the model by means of a hierarchical clustering approach similar

† Permanent address: Romanian Academy of Sciences, Centre for Mathematical Statistics, Bd. Magheru 22 70158 Bucuresti 22, Romania.

to that considered in the context of random processes with almost complete connections (Vlad 1992a).

2. Formulation of the model

Following Vlad (1991) we shall consider a branched process such as the growth of a population or a chemical or nuclear chain reaction

$$\ldots + X \rightarrow \nu X + \ldots, \nu = 0, 1, 2, \ldots. \tag{1}$$

Although the processes (1) are not necessarily related to chemistry, for the sake of simplicity we shall use the chemical terminology. Thus, we shall refer to the individual X as an 'active intermediate' and to reactions (1) with $\nu=0$, $\nu=1$, $\nu>1$ as the termination, propagation and branching processes, respectively.

To each generation q of intermediates we shall attach a probability λ_q that an individual is generated and a probability μ_q that the branching process does not terminate at the qth generation. In terms of these probabilities we can evaluate the following probability distributions:

(a) the probability $p_q(\nu)$ that at qth generation the offspring of an individual from the $(q-1)$th generation is ν:

$$p_q(\nu) = \lambda_q^\nu (1-\lambda_q) \tag{2}$$

(b) the probability φ_q that the process stops after q generations:

$$\varphi_q = \mu_1 \ldots \mu_q (1-\mu_{q+1}); \varphi_0 = 1-\mu_1. \tag{3}$$

Our aim is that, starting from λ_q and μ_q, to compute the probability $\xi_q(n)$ that at the qth generation the number of offspring of an individual from the 0th generation is n and the probability

$$\xi(n) = \sum_{q=0}^{\infty} \varphi_q \xi_q(n) \tag{4}$$

that the offspring from all generations is n. We note that the probability $\xi(n)$ was introduced by Vlad (1991) and that in a certain sense it expresses a kind or renormalization-like transformation.

At least in principle, $\xi_q(n)$ and $\xi(n)$ can be evaluated by applying the theory of stochastic branching processes (Athreya and Ney 1972, Harris 1989). According to this theory, the generating function

$$G_q(z)=\sum_{n=0}^{\infty}\xi_q(n)z^n \tag{5}$$

of $\xi_q(n)$ is the qth functional iterate of the generating function

$$f_q(z)=\sum z^\nu p_q(\nu). \tag{6}$$

We have

$$G_q(z)=f^{(*q)}(z) \tag{7}$$

with

$$f^{(*q)}(z)=f(f^{(*(q-1))}(z)) \qquad f^{(*0)}(z)=z. \tag{8}$$

In the case

$$\lambda_q=\lambda=\text{independent of } q \tag{9a}$$

$$\mu_q=\mu=\text{independent of } q \tag{9b}$$

the evaluation of $\xi(n)$ was reduced by Vlad (1991) to a problem considered in the literature. We note that in this case $p_q(\nu)=\lambda^\nu(1-\lambda)$ is given by the same Pascal law for all generations. This law corresponds to the problem of stimulated absorbtion of photons in an infinite medium (van Vliet and Zijlstra 1977; a similar law had been used in demography long time before, Lotka 1939).

Here our purpose is different. As we are mainly interested in the analysis of the memory effects at the critical state, we have to remove the restriction (9*a*). Indeed, at the critical state a kind of self-similarity on the whole is valid and the probability of the chain process termination $1-\mu_q$ may be assumed to be the same for all generations; however the memory yields to a dependence of the particle generation rate on the generation index q and thus λ_q should be q-dependent. By solving the problem of functional iteration for distinct λ_q we obtain (see appendix 1)

$$\xi_q(n)=(1-\delta_{no})\frac{g_q}{(1+\eta_q)^2}\left(\frac{\eta_q}{1+\eta_q}\right)^{n-1}+\delta_{no}\frac{1+\eta_{q-1}}{1+\eta_q} \tag{10}$$

where

$$g_q=\prod_{q'=1}^{q}[\lambda_{q'}/(1-\lambda_{q'})] \tag{11}$$

and

$$\eta_q=\sum_{q'=1}^{q}g_{q'}. \tag{12}$$

In order to evaluate the behaviour of $\xi(n)$ more information about the dependence of λ_q on q is necessary.

3. Non-critical regimes

Far from the critical state the memory effects are missing and we can assume that $\lambda_q = \lambda =$ independent of q. We get

$$\xi_q(n) = \bar{\nu}^q \left(\frac{1-\bar{\nu}}{1-\bar{\nu}^{q+1}}\right)^2 \left[1 - \frac{1-\bar{\nu}}{1-\bar{\nu}^{q+1}}\right]^{n-1} (1-\delta_{no}) + \delta_{no} \frac{1-\bar{\nu}^q}{1-\bar{\nu}^{q+1}} \tag{13}$$

where

$$\bar{\nu} = \sum_{\nu=0}^{\infty} \nu p_q(\nu) = \lambda/(1-\lambda) \tag{14}$$

is the average offspring of an individual per generation. Depending on the value of $\bar{\nu}$ we shall distinguish the following cases.

(a) *The supracritical regime*

$$\bar{\nu} > 1. \tag{15}$$

The growth of the population is explosive. Although at each step a finite probability of extinction $\xi_q(0) = (1-\bar{\nu}^q)/(1-\bar{\nu}^{q+1})$ exists as $q \to \infty$ all moments of n tend to infinity. Even if $\bar{\nu}$ is not very large, for sufficiently large q we have

$$\bar{\nu}^q \gg 1 \tag{16}$$

and $\xi_q(n)$ and $\xi(n)$ may be expressed as

$$\xi_q(n) \cong \bar{\nu}^{-q} \exp(-n\bar{\nu}^{-q}) \qquad q \to \infty \qquad n \to \infty \tag{17a}$$

$$\xi(n) \cong \sum_{q=0}^{\infty} (1-\mu)(\mu/\bar{\nu})^q \exp(-n\bar{\nu}^{-q}) \qquad n \to \infty. \tag{17b}$$

Equation (17*b*) has a self-similar form. By evaluating its asymptotic behaviour by means of the Poisson summation formula we come to (see appendix 2):

$$\xi(n) \sim n^{-(1+\mathcal{H})} \Xi(\ln n) \qquad \text{as } n \to \infty \tag{18}$$

where the fractal exponent $\mathcal{H}$ depends on $\bar{\nu}$ and μ:

$$\mathcal{H} = \ln(1/\mu)/\ln \bar{\nu} \tag{19}$$

$\Xi(\ln n)$ is a periodic function of $\ln n$ with period $\ln \bar{\nu}$:

$$\Xi(\ln n) = \frac{1-\mu}{\ln \bar{\nu}} \left\{ \Gamma(1+\mathcal{H}) + 2\sum_{m=1}^{\infty} \left[F^+(1+\mathcal{H}, 2\pi m/(\ln \bar{\nu})) \cos[2\pi m(\ln n)/(\ln \bar{\nu})] + F^-(1+\mathcal{H}, 2\pi m/(\ln \bar{\nu})) \sin[2\pi m(\ln n)/(\ln \bar{\nu})] \right] \right\} \tag{20}$$

where $\Gamma(x)$ is Euler's gamma function

$$\Gamma(x) = \int_0^{\infty} y^{x-1} \exp(-y)\, dy. \tag{21}$$

and $F^{\pm}(a, b)$ are the real and imaginary parts of the gamma function of complex argument, respectively

$$F^{\pm}(a,b)=\begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\Gamma(z=a+ib)=\int_0^\infty y^{a-1}\,\mathrm{e}^{-y}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(b\ln y)\,\mathrm{d}y. \qquad (22)$$

Scaling equations similar to (18) have been derived by different authors in different physical contexts (Novikov 1966, Shlesinger and West 1991 and West 1990 and references therein).

(b) *The sub-critical regime*

$$\bar{\nu}<1. \qquad (23)$$

In this case the population decreases from generation to generation and eventually becomes extinct. We have the following expressions for the asymptotic behaviour:

$$\xi_q(n)\cong\bar{\nu}^{q-1}(1-\bar{\nu})^2\exp[-n\ln(1/\bar{\nu})] \qquad n\to\infty \qquad q\to\infty \qquad (24a)$$

$$\xi(n)\cong\bar{\nu}^{-1}(1-\mu)(1-\mu\bar{\nu})^{-1}(1-\bar{\nu})^2\exp[-n\ln(1/\bar{\nu})] \qquad \text{as } n\to\infty. \qquad (24b)$$

4. The long memory effects

First of all we shall investigate the critical regime by neglecting the memory effects, that is by assuming that $\lambda_q=\lambda=$ independent of q and $\bar{\nu}=\lambda/(1-\lambda)=1$ and thus $\lambda=\frac{1}{2}$. We have

$$\xi_q(n)=(1-\delta_{n0})(1+q)^{-2}[q/(1+q)]^{n-1}+\delta_{n0}[q/(1+q)]. \qquad (25)$$

We note some similarities with the supracritical regime. As $q\to\infty$ the moments of n also increase towards infinity; however, the increase is slower than in the supracritical case. The asymptotic behaviour of $\xi_q(n)$ is expressed by a relationship similar to (17a)

$$\xi_q(n)\cong(1+q)^{-2}\exp[-n/(1+q)] \qquad q\to\infty \qquad n\to\infty. \qquad (26)$$

The main difference is related to the behaviour of $\bar{\xi}(n)$:

$$\bar{\xi}(n)\cong(1-\mu)\sum_{q=0}^{\infty}\mu^q(1+q)^{-2}\exp[-n/(1+q)] \qquad \text{as } n\to\infty. \qquad (27)$$

Here a difficulty arises in the evaluation of $\bar{\xi}(n)$ as $n\to\infty$. Unfortunately the Poisson formula cannot be applied to (27); however, noticing that the magnitude of a term in the series (27) is the result of multiplication of an exponentially decreasing function of q (μ^q) and of an increasing function of q ($\exp[-n/(1+q)]$) we can evaluate the value of $\bar{\xi}(n)$ by means of the method of steepest descent. After some routine manipulations we get

$$\bar{\xi}(n)\sim\text{constant}\; n^{-1}\exp(-K(0)n^{1/2}) \qquad (28a)$$

where

$$K(0)=2(\ln(1/\mu))^{1/2}. \qquad (28b)$$

The method of steepest descent cannot be used to evaluate the value of the pre-exponential constant.

In order to incorporate memory into the model we observe that starting from the sub-critical regime the critical state can be reached by considering that

$$\nu=\lambda/(1-\lambda))\leqslant 1 \qquad \nu\rightarrow 1 \tag{29a}$$

i.e.

$$\lambda\leqslant 1/2 \qquad \lambda\rightarrow 1/2. \tag{29b}$$

Let us assume that the limit (29*a*–*b*) occurs from generation to generation, that is, we start from a value $\lambda_0<\frac{1}{2}$ and through a sequence $\lambda_0, \lambda_1, \lambda_2, \ldots$ we eventually reach the critical value $\lambda_q\rightarrow\frac{1}{2}$ as $q\rightarrow\infty$. This limit procedure can be expressed in terms of a 'relative probability'

$$\varepsilon_q=\lambda_q/(1-\lambda_q)\leqslant 1 \qquad \text{for } \lambda_q\leqslant 1/2. \tag{30}$$

For $\lambda_q\leqslant\frac{1}{2}$, ε_q can be interpreted as the probability that a particle from the $(q-1)$th generation is replaced by another particle from the qth generation, provided that a termination process (equation (1) for $\nu=0$) does not take place. An alternative interpretation is related to the memory loss. Making an analogy with the work of Vlad (1992a) we shall assume that the memory loss occurs in a hierarchical manner, i.e. that the individual events of memory loss are lumped into blocks, the blocks into blocks of blocks, etc. The number l of successive lumping events generating a certain block is iteself random. Denoting by c the probability that the memory is lost after a lumping step ε_q can be evaluated as an average over all possible values of the lumping events

$$\varepsilon_q=\sum_{l=1}^{\infty} c^l\chi_l(q) \tag{31}$$

where $\chi_l(q)$ is the probability of occurrence of l lumping events. Denoting by b the probability that a lumping event corresponds to an elementary process from a given block, we have

$$\chi_l(q)=(1-b^q)(b^q)^{l-1}. \tag{32}$$

By evaluating the sum (31) we get

$$\varepsilon_q(c, b)=c(1-b^q)/(1-cb^q). \tag{33}$$

As finally the whole memory is lost we consider the limit $c, b\rightarrow 1$. Following Vlad (1992a), we shall assume that the ratio

$$H=\ln c/\ln b=\text{constant}\leqslant 1. \tag{34}$$

H is a kind of 'fractal exponent' attached to the lumping process. We get

$$\varepsilon_q=q/(q+H) \qquad \text{i.e. } \lambda_q=q/(2q+H) \tag{35}$$

and therefore

$$g_q=q!\Gamma(1+H)/\Gamma(H+q+1) \tag{36}$$

and

$$\eta_q=\sum_{q'=1}^{q} q'!\Gamma(1+H)/\Gamma(H+q'+1). \tag{37}$$

By taking equations (4), (10) and (37) into account and applying the method of steepest descent we get the following expressions for the asymptotic behaviour of $\xi_q(n)$ and $\bar{\xi}(n)$:

$$\xi_q(n) \cong \frac{(1-H)^2}{\Gamma(1+H)} q^{-(2-H)} \exp\left(-n \frac{1-H}{\Gamma(1+H)} q^{-(1-H)}\right) \quad \text{as } q \to \infty \qquad n \to \infty \tag{38}$$

and

$$\bar{\xi}(n) \sim \text{constant } n^{-1} \exp[-K(H) n^{1/(2-H)}] \qquad \text{as } n \to \infty \tag{39}$$

where

$$K(H) = (1-H)^{H/(2-H)}(2-H)[\ln(1/\mu)]^{(1-H)/(2-H)}/[\Gamma(1+H)^{1/(2-H)}]. \tag{40}$$

The complete memory loss corresponds to $H=0$. In this case (39)–(40) reduce to (28)–(29). In contrast, for $H \to 1$ the memory effects are very strong and even at the critical state for $n \to \infty$, $\bar{\xi}(n)$ tends towards an exponential.

5. Moments

A referee raised the problem of physical significance of the critical regime. In order to answer this question we shall evaluate the factorial moments

$$\Lambda_l(q) = \sum_{n=0}^{\infty} \xi_q(n) n(n-1) \ldots (n-l+1) \tag{41}$$

$$\bar{\Lambda}_l = \sum_{n=0}^{\infty} \bar{\xi}(n) n(n-1) \ldots (n-l+1) \tag{42}$$

of the number of particles from the qth generation and of the total offspring number, respectively. If the memory effects are missing these factorial moments can be easily evaluated by applying the generating function technique, resulting in:

(a) *The sub-critical regime*

$$\Lambda_l(q) = l! \bar{\nu}^q \left[\frac{\bar{\nu}(1-\bar{\nu}^q)}{1-\bar{\nu}}\right]^{l-1} \qquad \bar{\nu} < 1 \tag{43}$$

$$\bar{\Lambda}_l = l[(l-1)!]^2 \left(\frac{\bar{\nu}}{1-\bar{\nu}}\right)^{l-1} \sum_{k=0}^{l-1} \frac{(-1)^k}{k!(l-1-k)!} \frac{1-\mu}{1-\mu\bar{\nu}^{k+1}} \qquad \bar{\nu} < 1. \tag{44}$$

(b) *The critical regime*

$$\Lambda_l(q) = l! q^{l-1} \qquad \bar{\nu} = 1, \tag{45}$$

$$\bar{\Lambda}_l = l! \sum_{m=1}^{l-1} \$_{l-1}^{(m)} \left(\frac{\mu}{1-\mu}\right)^m m! \qquad \bar{\nu} = 1 \tag{46}$$

where

$$\$_{l-1}^{(m)} = \sum_{k=1}^{m} (-1)^{m-k} \frac{k^{l-1}}{k!(m-k)!} \tag{47}$$

are the Stirling numbers of the second kind.

(c) *The supracritical regime*

$$\Lambda_l(q)=l!\bar{\nu}^q\left[\frac{\bar{\nu}(\bar{\nu}^q-1)}{\bar{\nu}-1}\right]^{l-1} \qquad \bar{\nu}>1 \tag{48}$$

$$\tilde{\Lambda}_l=l[(l-1)!]^2\left(\frac{\bar{\nu}}{\bar{\nu}-1}\right)^{l-1}\sum_{k=0}^{l-1}\frac{(-1)^{l-1-k}}{k!(l-1-k)!}\cdot\frac{1-\mu}{1-\mu\bar{\nu}^{k+1}} \qquad \text{for } (1/\mu)^{1/l}>\bar{\nu}>1 \tag{49}$$

$$\tilde{\Lambda}_l=\infty \qquad \text{for } \bar{\nu}\geqslant(1/\mu)^{1/l}. \tag{50}$$

By examining (41)–(50) we see that the critical regime ($\bar{\nu}=1$) does not correspond to a gelation-like point. Such a point is characterized by a minimum value of $\bar{\nu}$ for which all moments $\tilde{\Lambda}_l$ of the total number of particles are infinite. From (49)–(50) we note that the gelation-like point corresponds to

$$\bar{\nu}=1/\mu \tag{51}$$

that is, to a particular case of the supracritical regime. From this point of view it is useful to make a distinction between two supracritical sub-regimes:

(1) *The pregelation regime*, characterized by

$$1/\mu>\bar{\nu}>1. \tag{52}$$

In this case the exponent $\mathcal{H}$ entering the asymptotical expression of $\xi(n)$ (equation (18)) is bigger than unity

$$\mathcal{H}=\ln(1/\mu)/\ln\bar{\nu}>1 \tag{53}$$

and only the moments $\tilde{\Lambda}_l$ having an index l equal or greater than $\mathcal{H}$ are infinite:

$$\tilde{\Lambda}_l=\text{finite} \qquad l<\mathcal{H}, \tag{54}$$

$$\tilde{\Lambda}_l=\infty \qquad l\geqslant\mathcal{H}. \tag{55}$$

(2) *The gelation and postgelation regime*, characterized by

$$\bar{\nu}\geqslant1/\mu. \tag{56}$$

We have

$$1\geqslant\mathcal{H}>0 \tag{56}$$

and all moments $\tilde{\Lambda}_l$ are infinite

$$\tilde{\Lambda}_l=\infty \qquad l=1,2,\ldots. \tag{58}$$

To clarify the nature of the critical point we shall evaluate from (43)–(50) the means and the dispersions of the numbers of particles corresponding to the different cases mentioned before. We get:

(a) *The sub-critical regime*

$$\langle n(q)\rangle=\bar{\nu}^q=\exp[-q\ln(1/\bar{\nu})] \qquad q\to\infty \tag{59}$$

$$\langle\Delta n^2(q)\rangle=\frac{2\bar{\nu}}{1-\bar{\nu}}\bar{\nu}^q(1-\bar{\nu}^q)\cong\frac{2\bar{\nu}}{1-\bar{\nu}}\exp[-q\ln(1/\bar{\nu})] \qquad q\to\infty \tag{60}$$

$$\langle\tilde{n}\rangle=(1-\mu)/(1-\mu\bar{\nu}) \tag{61}$$

$$\langle \Delta \bar{n}^2 \rangle = \frac{\mu(1-\mu)}{1-\mu\bar{\nu}} \left(\frac{2\bar{\nu}^2}{1-\mu\bar{\nu}^2} + \frac{1-\bar{\nu}}{1-\mu\bar{\nu}} \right). \tag{62}$$

We note that both $\langle n(q) \rangle$ *and* $\langle \Delta n^2(q) \rangle$ tend to 0 as $q \to \infty$ in the same way, resulting in the scaling equation

$$\langle \Delta n^2(q) \rangle \cong \frac{2\bar{\nu}}{1-\bar{\nu}} \langle n(q) \rangle \qquad \text{as } q \to \infty. \tag{63}$$

(b) *The critical regime*

$$\langle n(q) \rangle = 1 \tag{64}$$

$$\langle \Delta n^2(q) \rangle = 2q \tag{65}$$

$$\langle \bar{n} \rangle = 1 \tag{66}$$

$$\langle \Delta \bar{n}^2 \rangle = 2\mu/(1-\mu). \tag{67}$$

The equivalence between the behaviour of $\langle n(q) \rangle$ and $\langle \Delta n^2(q) \rangle$ as $q \to \infty$ no longer exists. Whereas the mean value $\langle n(q) \rangle$ is constant the dispersion $\langle \Delta n^2(q) \rangle$ tends to ∞ as $q \to \infty$. This fact shows that as $q \to \infty$ the number of particles from a generation has an intermittent behaviour.

(c) *The supracritical regime*

$$\langle n(q) \rangle = \bar{\nu}^q = \exp(q \ln(\bar{\nu})) \qquad q \to \infty \tag{68}$$

$$\langle \Delta n^2(q) \rangle = \frac{2\bar{\nu}}{\bar{\nu}-1} \bar{\nu}^q(\bar{\nu}^q - 1) \cong \frac{2\bar{\nu}}{\bar{\nu}-1} \exp[2q \ln \bar{\nu}] \qquad q \to \infty \tag{69}$$

$$\langle \bar{n} \rangle = (1-\mu)/(1-\mu\bar{\nu}) \qquad 1/\mu > \bar{\nu} > 1 \tag{70}$$

$$\langle \bar{n} \rangle = \infty \qquad \nu \geqslant 1/\mu \tag{71}$$

$$\langle \Delta \bar{n}^2 \rangle = \frac{\mu(1-\mu)}{1-\mu\bar{\nu}} \left(\frac{2\bar{\nu}^2}{1-\mu\bar{\nu}^2} - \frac{\bar{\nu}-1}{1-\mu\bar{\nu}} \right) \qquad (1/\mu)^{1/2} > \bar{\nu} > 1 \tag{72}$$

$$\langle \Delta \bar{n}^2 \rangle = \infty \qquad \bar{\nu} \geqslant (1/\mu)^{1/2}. \tag{73}$$

As $q \to \infty$ both $\langle n(q) \rangle$ and $\langle \Delta n^2(q) \rangle$ increase exponentially to ∞. The asymptotic rate of increase of $\langle \Delta n^2(q) \rangle$ is $2 \ln \bar{\nu}$, that is, two times bigger than the rate of increase $\ln \bar{\nu}$ of $\langle n(q) \rangle$. We have

$$\langle \Delta n^2(q) \rangle / \langle n(q) \rangle \cong \frac{2\bar{\nu}}{\bar{\nu}-1} \exp(q \ln \bar{\nu}) \qquad q \to \infty \tag{74}$$

$$\langle \Delta n^2(q) \rangle \cong \frac{2\bar{\nu}}{\bar{\nu}-1} \langle n(q) \rangle^2 \qquad q \to \infty. \tag{75}$$

From these equations we note that the intermittent behaviour is stronger than in the case of the critical regime.

From the above analysis it follows that $\bar{\nu} = 1$ is the minimum value of $\bar{\nu}$ for which the intermittent behaviour of $n(q)$ occurs. Thus the critical regime corresponds to the onset of intermittent behaviour for the number of particles from a generation.

The analysis of the memory effects is more difficult. The evaluation of some moments invoves the summation of certain series which cannot be computed exactly in the general case. However, the analysis of their asymptotic behaviour is still possible by means of analytical methods. We get the following expressions for the factorial moments:

$$\tilde{\Lambda}_l(q)=\frac{l!q!\Gamma(1+H)}{\Gamma(1+H+q)}\left(\sum_{q'=1}^{q}\frac{q'!\Gamma(1+H)}{\Gamma(1+H+q')}\right)^{l-1} \tag{76}$$

$$\tilde{\Lambda}_l=(1-\mu)\Gamma(1+H)l!\sum_{q=0}^{\infty}\frac{\mu^q q!}{\Gamma(q+H+1)}\left(\sum_{q'=1}^{q}\frac{q'!\Gamma(1+H)}{\Gamma(q'+H+1)}\right)^{l-1} \tag{77}$$

from which we can evaluate the asymptotic behaviour for $\langle n(q)\rangle$ and $\langle\Delta n^2(q)\rangle$:

$$\langle n(q)\rangle=\frac{q!\Gamma(1+H)}{\Gamma(q+H+1)}\cong\Gamma(1+H)q^{-H} \qquad \text{as } q\to\infty \tag{78}$$

$$\langle\Delta n^2(q)\rangle=\frac{q!\Gamma(1+H)}{\Gamma(q+H+1)}\left[2\sum_{q'=1}^{q}\frac{q'!\Gamma(1+H)}{\Gamma(q'+H+1)}+1-\frac{q!\Gamma(1+H)}{\Gamma(q+H+1)}\right]$$

$$\cong\{[2\Gamma^2(1+H)]/(1-H)\}q^{1-2H} \qquad \text{as } q\to\infty \tag{79}$$

and thus

$$\langle\Delta n^2(q)\rangle/\langle n(q)\rangle\cong\{[2\Gamma(1+H)]/(1-H)\}q^{1-H} \qquad \text{as } q\to\infty \tag{80}$$

$$\langle\Delta n^2(q)\rangle\cong\{2[\Gamma(1+H)]^{1/H}/(1-H)\}\langle n(q)\rangle^{(2H-1)/H} \qquad \text{as } q\to\infty. \tag{81}$$

For $q\to\infty$ the mean value $\langle n(q)\rangle$ decreases to 0 as q^{-H}. The evolution of $\langle\Delta n^2(q)\rangle$ depends on the value of the exponent H. For $H<\frac{1}{2}$ it increases to ∞, for $H=\frac{1}{2}$ it tends towards a constant value and for $H>\frac{1}{2}$ it decreases to 0. Although less pronounced than in the case when memory is missing ($H=0$) the intermittent behaviour of $n(q)$ still exists. Indeed, the ratio $\langle\Delta n^2(q)\rangle/\langle n(q)\rangle$ increases to ∞ as q^{1-H} for $q\to\infty$ for any $1>H\geqslant 0$. For $H>0$ the increase is slower than the q-dependence given by (64)–(65) for the case when the memory is missing.

The mean value $\langle\tilde{n}\rangle$ of the total offspring number can be expressed in a closed form

$$\langle\tilde{n}\rangle=[(1-\mu)/\mu]^H B(1+H,1-H,\mu)+1-\mu \tag{82}$$

where

$$B(p,q,x)=\int_0^x x^{p-1}(1-x)^{q-1}\,dx \tag{83}$$

is the incomplete beta function. The asymptotic behaviour of $\langle\tilde{n}\rangle$ for $\mu\to 0$ and $\mu\to 1$ is given by

$$\langle\tilde{n}\rangle\cong 1-\frac{H}{H+1}\mu+0(\mu^2) \qquad \text{as } \mu\to 0 \tag{84}$$

and

$$\langle\tilde{n}\rangle\cong(1-\mu)^H\frac{\pi H}{\sin(\pi H)} \qquad \text{as } \mu\to 1. \tag{85}$$

On physical grounds we expect that all moments of the total offspring number exist and are finite for any value of the memory exponent $1>H\geqslant 0$. Indeed the memory effects lead to a slower increase for the first generations whereas for the last generations the rate is the same as in the case when the memory effects are missing. To prove this we shall evaluate the parameter $\bar{\nu}$ for $1>H\geqslant 0$. From (2) and (35) we have

$$\bar{\nu}_q(H)=\sum_{\nu=0}^{\infty}\nu\left(1-\frac{q}{2q+H}\right)\left(\frac{q}{2q+H}\right)^{\nu}=\frac{q}{q+H}\leqslant 1. \tag{86}$$

i.e. for $q<\infty$, $\bar{\nu}_q(H)$ is smaller than the value $\bar{\nu}=1$ corresponding to $H=0$, whereas for $q\to\infty$ it tends to the asymptotic value $\bar{\nu}=1$. Thus the increase of the number of particles for $H>0$ should be smaller than the one corresponding to $H=0$. As the moments corresponding to the critical regime without memory ($H=0$) exist and are finite (see equation (46)), the moments corresponding to $H>0$ should be also finite. By examining (77) it is easy to see that for $\mu<1$ this is indeed the case. All terms in the series (77) are positive and tend to 0 as

$$q^{l(1-H)-1}\mu^q\to 0 \qquad \text{for } q\to\infty. \tag{87}$$

The exponential decrease to 0 given by (87) ensures the convergence of the series (77).

6. Discussion

Within this paper we have generalized a model developed by Vlad (1991) for supracritical branched chain processes by incorporating the memory effects typical of the critical state. The memory is assumed to be described in terms of a hierarchical lumping process characterized by a certain fractal exponent.

The asymptotic behaviour for the probability distribution $\xi(n)$ of the total offspring of an initial particle depends on the probabilities λ_q that a new particle is generated at the qth generation. The subcritical regime corresponds to $\frac{1}{2}>\lambda_q=\lambda=$ independent of q. In this case $\xi(n)$ has an exponential tail. As λ_q increase the memory effects become important and λ_q are q-dependent with $\lambda_q\to\frac{1}{2}$ as $q\to\infty$. This is the critical regime. The corresponding tail of $\xi(n)$ is a streched exponential constant $n^{-1}\exp[-K(H)n^{\alpha}]$ with an exponent $\alpha=1/(2-H)$ between $\frac{1}{2}$ ($H=0$, no memory) and $\alpha\to 1$ ($H\to 1$, very strong memory). A supercritical regime is reached when λ_q become bigger than $\frac{1}{2}$ and again independent of q. For this regime $\xi(n)$ is characterized by an inverse power tail in n modulated by a periodic function in $\ln n$.

In the subcritical regime all moments of the number of particles exist and are finite. In particular, as the number q of generations tends to infinity, $q\to\infty$, both the mean value and the dispersion of the number of particles from a generation decay to 0 with the same rate. Although in the critical regime the moments are still finite, an intermittent behaviour of the numbers of particles from a generation emerges whether the memory is present or not. The memory generates a slowing down of the growth process resulting in an unexpected behaviour of the average value and dispersion: the average number of particles from a generation slowly decreases with increase of the generation index; depending on the value of the memory exponent H, the corresponding dispersion either increases or decreases. In the first part of the supracritical regime, corresponding to $1/\mu>\bar{\nu}>1$, the intermittent behaviour is very strong. In particular the dispersion of the number of particles from a given generation exponentially increases with a rate which is two times bigger than the rate of increase of the average value. For $1/\mu > \bar{\nu} > 1$ the superior moments of the total offspring number are infinite. As the system is approaching the 'gelation point' $\bar{\nu} \to 1/\mu$ the first moments also become infinite. Eventually for $\bar{\nu} \geq 1/\mu$ all moments are infinite and there is no characteristic scale for the total offspring number.

The model presented in this paper may be applied to a broad class of natural phenomena. For instance, the process (1) may represent the growth of a branched polymer, a chemical explosion in gases, fragmentation dynamics or the growth of a population. However, not all these processes display all regimes discussed before. A necessary condition for the existence of all the three regimes is the interplay between the generation and decay processes of the X particles: the decay processes lead to the possibility that the final offspring number of an inital particle is equal to 0; on the contrary the branched generation processes lead to the possibility of an explosive increase of the number of particles. The equilibration between these two opposite factors leads to the three regimes discussed before. For example a chemical chain reaction in gases, the growth of a branched polymer or of a population belong to this class of processes.

A fragmentation process with mass conservation has a different behaviour. Due to mass conservation it is impossible that the final offspring number is 0. The system should contain at least one particle which is made up of the total amount of matter initially present in the system. The theory may be also applied to this kind of process; in this case one or two of the characteristic regimes described before are missing.

After submitting this paper for publication we learnt about two recent articles by Berlin *et al* (1992a, b) which also deal with memory effects for branched chain processes. Their approach is related to the growth of biological populations rather than to the physico-chemical processes considered here. Memory effects are introduced by assuming that the fertility of an individual is an inherited feature. They discuss the time dependence of the moments of population size rather than the probability distribution of the final offspring number originating from an individual. That is why a comparison between the two approaches is not easy. At present it is not clear whether the approach of Berlin *et al* (1992a, b) is also of interest in connection with certain physico-chemical processes or not.

Let us outline some open questions and limitations of our approach. A first objection is related to the fact that no specific predictions concerning the values of the probabilities c and b or of the fractal exponent H are made. Another limitation of the theory is related to the impossibility of analytical evaluation of the pre-exponential coefficients in (28) and (39). It might also be possible that these coefficients are in fact slowly varying functions of n. The testing of this assumption is impossible within the framework of the steepest descent approximation.

On the other hand, describing the memory in terms of a hierarchical lumping process is rather obscure. Broadly speaking the absence of the memory corresponds to a set of q-independent probabilities $\lambda_1 = \lambda_2 = \ldots = \lambda =$ independent of q. An appropriate model should be built up by explaining how starting from a memoryless system the memory effects are generated. In contrast, we have derived our model in a rather arbitrary way by analysing how the memory could be lost. Because of this drawback our picture is rather incomplete. In particular no clear physical explanation for the mechanism of memory action is suggested. It might be possible that a more detailed model of memory action could be derived by combining the above-presented approach with a new hierarchical clustering approach to stochastic renormalisation

(Vlad 1992b). This is a separate project which is planned to be the subject of future research.

Acknowledgments

The author wishes to thank professor K Kehr for helpful suggestions, the Alexander von Humboldt Foundation for financial support and the Forschungszentrum Jülich for their hospitality. Thanks are also due to professor J Ross for his hospitality at Stanford University, where the final version of this paper was written.

Appendix 1

The generating function $f_q(z)$ corresponding to (2) is

$$f_q(z)=(1-\lambda_q)/(1-z\lambda_q) \tag{A1.1}$$

so that the solving of the functional iteration (8) reduces to the solving of a nonlinear difference equation with variable coeefficients

$$f^{(*q)}-\lambda_q f^{(*q)} f^{(*(q-1))}=1-\lambda_q. \tag{A1.2}$$

Through the substitution

$$f^{(*q)}=A_q/A_{q+1} \tag{A1.3}$$

(A1.2) reduces to a second-order linear difference equation

$$(1-\lambda_q)A_{q+1}-A_q+\lambda_q A_{q-1}=0. \tag{A1.4}$$

By solving (A1.4) with the initial conditions

$$A_0=z \qquad A_1=1 \tag{A1.5}$$

which corrrespond to $f^{(*0)}(z)=z$, coming back to $f^{(*q)}(z)$ and using (7) we get an analytical expression for $G_q(z)$

$$G_q(z)=[1+\eta_{q-1}(1-z)]/[1+\eta_q(1-z)]. \tag{A1.6}$$

By developing (A1.6) in a Taylor series in z and comparing the result with (5) we come to (10).

Appendix 2

The sum (17*b*) may be evaluated by means of the Poisson summation formula (West 1990):

$$\sum_{q=0}^{\infty} h(q)=(1/2)h(0)+\int_0^{\infty} h(x)\,\mathrm{d}x+2\sum_{m=1}^{\infty}\int_0^{\infty} h(x)\cos(2\pi mx)\,\mathrm{d}x \tag{A2.1}$$

where

$$h(x)=(\mu/\bar{\nu})^x \exp(-n\bar{\nu}^{-x}). \tag{A2.2}$$

By combining (17*b*) and (A2.1)–(A2.2) and making the change of variable

$$y=n\bar{\nu}^{-x} \tag{A2.3}$$

we can express $\xi(n)$ as a sum of integrals over y from 0 to n. These integrals have the structure of incomplete gamma functions of different complex arguments. As $n \to \infty$ they converge towards the corresponding complete gamma functions. By replacing the integration limit n by ∞ after some arrangements we obtain (18)–(20).

References

Athreya K G and Ney P 1972 *Branching Processes* (Berlin: Springer)
Berlin Y A, Drobnitsky D O, Goldanskii V I and Kuz'min V V 1992a *BioSystems* **26** 185
Berlin Y A, Drobnitsky D O, Goldanskii V I and Kuz'min V V 1992b *Phys. Rev.* A **45** 3547
Bouchaud J P and Georges A 1990 *Phys. Rep.* **195** 127
Freed K F 1987 *Renormalization Group Theory of Macromolecules* (New York: John Wiley)
Harris T E 1989 *The Theory of Branching Processes* (New York: Dover)
Haus J W and Kehr K W 1978 *Solid State Commun.* **26** 753
Haus J W and Kehr K W 1979 *J. Phys. Chem. Solids* **40** 1019
Iosifescu M and Grigorescu S 1989 *Dependence with Complete Connections and Applications* (Cambridge: Cambridge Tracts)
Kutner R 1985 *J. Phys. C: Solid State Phys.* **18** 6323
Lotka A J 1939 *Théorie Analytique des Associations Biologiques* vol. II (Paris: Hermann)
Novikov E A 1966 *Sov. Phys. Dok.* **11** 497
Peliti L and Pietronero K 1987 *Riv. Nuovo Cimento* **10** 1
Pietronero L and Sibesma A P 1987 *Europhys. Lett.* **4** 869
Shlesinger M F and Klafter J 1989 *J. Phys. Chem.* **93** 7023
Shlesinger M F and West B J 1991 *Phys. Rev. Lett.* **67** 2106
Van Vliet K M and Zijlstra R J J 1977 *Physica* **89A** 353
Vlad M O 1991 *Chem. Phys. Lett.* **182** 595
Vlad M O 1992a *J. Phys. A: Math. Gen.* **25** 749
Vlad M O 1992b *Phys. Rev.* A **45** 3600
West B J 1990 *J. Opt. Soc. Am.* A **7** 1074